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# AN APPLICATION OF THE FOURIER TRANSFORM TO SECTIONS OF STAR BODIES

BY

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ABSTRACT

We express the volume of central hyperplane sections of star bodies in $R^n$ in terms of the Fourier transform of a power of the radial function, and apply this result to confirm the conjecture of Meyer and Pajor on the minimal volume of central sections of the unit balls of the spaces $\ell_p^n$ with $0 < p < 2$.

## 1. Introduction

Let $K$ be a centrally symmetric star body in $\mathbb{R}^n$. Then the function $x \to \|x\| = \min\{a > 0: x \in aK\}, x \in \mathbb{R}^n$ is continuous, homogeneous of degree 1 on $\mathbb{R}^n$, and $\|x\| = 0$ if and only if $x = 0$. Denote by $\Omega$ the Euclidean unit sphere in $\mathbb{R}^n$. For $\xi \in \Omega$, let $\xi^\perp = \{x \in \mathbb{R}^n: (x, \xi) = 0\}$ be the hyperplane passing through the origin and having $\xi$ as its normal vector. We prove that the $(n-1)$-dimensional volume of the section of $K$ by $\xi^\perp$ is equal (up to a constant) to the Fourier transform (in the sense of distributions) of the function $\|x\|^{-n+1}$ at the point $\xi$, i.e. for every $\xi \in \Omega$,

$$(1) \qquad \mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)} (\|x\|^{-n+1})^\wedge(\xi).$$

The Fourier transforms of powers of the norms of the spaces $\ell_p^n$ have been calculated in [5] (for $0 < p < \infty$) and [6] (for $p = \infty$). In view of the formula (1), one can use these calculations to obtain formulae for the volume of

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central hyperplane sections. For example, the calculation from [5] allows us to extend (from $1 \le p \le 2$ to the case of arbitrary $p > 0$) the formula of Meyer and Pajor [7] for the volume of hyperplane sections of the unit ball $B_p = \{x \in \mathbb{R}^n : |x_1|^p + \cdots + |x_n|^p \le 1\}$ of the space $\ell_p^n$. For every $p > 0$ and $\xi \in \Omega$,

$$\mathrm{Vol}_{n-1}(B_p \cap \xi^\perp) = \frac{p}{\pi(n-1)\Gamma((n-1)/p)} \int_0^\infty \prod_{k=1}^n \gamma_p(t\xi_k)\,dt, \tag{2}$$

where $\gamma_p$ is the Fourier transform of the function $z \to \exp(-|z|^p), z \in \mathbb{R}$.

The formula (2) was proved in 1988 by Meyer and Pajor [7] for $1 \le p \le 2$ using a probabilistic argument. The general idea of formula (1) and the possibility of extending formula (2) to all $p > 0$ became clear to the author immediately after comparing the calculation from [5] with the result of [7]. When $p \to \infty$, (2) turns into the formula that was used by Ball [1] to prove his famous result on maximal sections of the unit cube in $\mathbb{R}^n$.

Meyer and Pajor [7] used formula (2) to show that the minimal section of the unit ball of the space $\ell_1^n$ is the one perpendicular to the vector $(1, 1, \ldots, 1)$, and the maximal section is perpendicular to the vector $(1, 0, \ldots, 0)$. They also showed that, for the unit balls of the spaces $\ell_p^n$ with $1 < p < 2$, the upper bound occurs in the same direction as for $p = 1$, and raised the question of whether, for every $p \in (1, 2)$, the minimal section has the same direction as in the case $p = 1$. Meyer and Pajor mentioned in their paper (without proof) that the answer to the question would be affirmative if one showed that the function $\gamma_p(\sqrt{t})$ is log-convex on $[0, \infty)$. In Lemma 3 below, we prove the log-convexity of the function $\gamma_p(\sqrt{t})$, which allows us to confirm the conjecture of Meyer and Pajor, i.e. for every $p \in (0, 2)$ the minimal section of $B_p$ is the one perpendicular to the vector $(1, 1, \ldots, 1)$.

We refer the reader to [2] for more information about sections of convex bodies.

## 2. The Fourier transform formula for sections of star bodies

As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $(\hat{f}, \hat{\phi}) = (2\pi)^n (f, \phi)$ for every test function $\phi$.

We need the following simple property of the Fourier transform of homogeneous functions of degree $-n+1$ on $\mathbb{R}^n$.

LEMMA 1: *Let $f$ be an even continuous homogeneous function of degree $-n+1$ on $\mathbb{R}^n \setminus \{0\}$, $n > 1$. Then, for every $\xi \in \Omega$,*

$$\hat{f}(\xi) = \pi \int_{\Omega \cap \{(\theta,\xi)=0\}} f(\theta) d\theta.$$

*Proof:* Because of the well-known connection between the Fourier transform and the Radon transform (see [4]), for every even test function $\phi$ and every $\theta \in \Omega$, the Fourier transform of the function $t \to \hat{\phi}(t\theta)$ at zero is equal to $\int_{\mathbb{R}} \hat{\phi}(t\theta)dt = 2\pi \int_{(\theta,\xi)=0} \phi(\xi)d\xi$. Also the Fourier transform of the $\delta$-function (defined by $(\delta, \phi) = \phi(0)$) is the constant function $h(t) = 1$. Therefore, passing to spherical coordinates and using homogeneity of $f$, we get

$$(\hat{f}, \phi) = \int_{\mathbb{R}^n} f(x)\hat{\phi}(x)dx = \int_{\Omega}\int_0^{\infty} f(t\xi)t^{n-1}\hat{\phi}(t\xi)dtd\xi$$
$$= (1/2)\int_{\Omega} f(\theta)d\theta \int_{\mathbb{R}} \hat{\phi}(t\theta)dt = \pi \int_{\Omega} f(\theta)d\theta \int_{(\theta,\xi)=0} \phi(\xi)d\xi$$
$$= \pi \int_{\mathbb{R}^n} \|\xi\|_2^{-1}\Big(\int_{\Omega\cap\{(\theta,\xi)=0\}} f(\theta)d\theta\Big)\phi(\xi)d\xi,$$

where $\|\cdot\|_2$ is the Euclidean norm, and the last equality follows from self-adjointness of the spherical Radon transform (see [2, p. 328]) and the fact that $\xi \mapsto \int_{(\theta,\xi)=0} \phi(\xi)d\xi$ is a homogeneous function on $\mathbb{R}^n$ of degree $-1$. Since $\phi$ is an arbitrary even test function, the result follows. ∎

Let $K$ be a centrally symmetric star body in $\mathbb{R}^n$. It is easily seen that, for every $\xi$ in the unit sphere $\Omega$, the $(n-1)$-dimensional volume of the section of $K$ by the hyperplane $\xi^{\perp} = \{x: (x, \xi) = 0\}$ satisfies the equality

$$\frac{\mathrm{Vol}_{n-1}(K \cap \xi^{\perp})}{\mathrm{Vol}_{n-1}(B_{n-1})} = \frac{\int_{\Omega\cap\xi^{\perp}} \|x\|^{-n+1}dx}{A_{n-1}}, \tag{3}$$

where $\mathrm{Vol}_{n-1}(B_{n-1}) = \pi^{(n-1)/2}/\Gamma((n+1)/2)$ is the volume of the Euclidean unit ball $B_{n-1}$ in $\mathbb{R}^{n-1}$, and $A_{n-1} = 2\pi^{(n-1)/2}/\Gamma((n-1)/2))$ is the surface area of the Euclidean unit sphere in $\mathbb{R}^{n-1}$.

The integral in the right-hand side of (3) is equal to the integral in Lemma 1 with $f(x) = \|x\|^{-n+1}$. Therefore, Lemma 1 and (3) imply the following Fourier transform formula for the volume of central sections of $K$:

THEOREM 1: *For every $\xi \in \Omega$,*

$$\mathrm{Vol}_{n-1}(K \cap \xi^{\perp}) = \frac{1}{\pi(n-1)}(\|x\|^{-n+1})^{\wedge}(\xi).$$

The Fourier transform of the functions of the form $f(\|x\|_\infty)$ was calculated in [6], where $\|x\|_\infty$ stands for the norm of the space $\ell_\infty^n$, and $f$ belongs to a large class of functions on $\mathbb{R}$. (Note that a multiplier $(-1)^{n-1}$ is missing in the formula in [6].) If we apply the formula from [6] to the function $f(t) = |t|^p$ with $p \in (-1, 0)$ (use the formulae for the Fourier transform of the functions $|t|^p$ and $|t|^p \operatorname{sgn}(t)$ from [3, p. 173]), then use analytic extension by $p$ and put $p = -n+1$, we get an expression for the Fourier transform of $\|x\|_\infty^{-n+1}$. For every $\xi \in \mathbb{R}^n$ with non-zero coordinates, if the dimension $n$ is odd we have

$$(\|x\|_\infty^{-n+1})^\wedge(\xi) =$$
$$\frac{(-1)^{(n-1)/2} 2^{-n+1} \sqrt{\pi}\Gamma((-n+2)/2)}{\Gamma((n-1)/2)\prod_{k=1}^n \xi_k} \sum_\delta \delta_1 \cdots \delta_n \Big|\sum_{j=1}^n \delta_j \xi_j\Big|^{n-1} \operatorname{sgn}(\sum_{j=1}^n \delta_j \xi_j).$$

If the dimension $n$ is even we have

$$(\|x\|_\infty^{-n+1})^\wedge(\xi) = \frac{(-1)^{(n-2)/2} 2^{-n+1} \sqrt{\pi}\Gamma((-n+3)/2)}{\Gamma(n/2)\prod_{k=1}^n \xi_k} \sum_\delta \delta_1 \cdots \delta_n \Big|\sum_{j=1}^n \delta_j \xi_j\Big|^{n-1}.$$

The outer sum is taken over all changes of signs $\delta = (\delta_1, \ldots, \delta_n), \delta_j = \pm 1$. Note that the coefficient at the outer sum is equal to $1/(2(n-1)!)$ in both odd and even cases. These formulae, in conjunction with Theorem 1, provide simple expressions for the volume of central sections of the cube $[-1,1]^n$. Previously, similar formulae were obtained using probabilistic arguments specifically designed for the cube. As mentioned in the introduction, Ball used those formulae in his paper [1] on the exact lower and upper bounds for the volume of central sections of the unit ball of the space $\ell_\infty^n$, which are equal to $2^{n-1}$ and $2^{n-1}\sqrt{2}$, respectively.

In order to prove formula (2), we compute the Fourier transform of the functions $\|x\|_p^\beta$, where $\|x\|_p$ stands for the norm of the space $\ell_p^n$.

Let us recall some properties of the functions $\gamma_p$ defined in the introduction. Firstly, for $0 < p \leq 2$, $\gamma_p$ is (up to a constant) the density of the standard $p$-stable measure on $\mathbb{R}$, so $\gamma_p$ is a non-negative function. For every $p > 0$,

$$\lim_{t\to\infty} t^{1+p}\gamma_p(t) = 2\Gamma(p+1)\sin(\pi p/2),$$

so $\gamma_p$ decreases at infinity as $|t|^{-1-p}$ (see [8]). Simple calculations show that $\gamma_p(0) = 2\Gamma(1+1/p)$, and $\int_0^\infty \gamma_p(t)dt = \pi$. The following calculation is taken from [5].

LEMMA 2: *Let $p > 0$, $n \in \mathbb{N}$, $-n < \beta < pn$, $\beta/p \notin \mathbb{N}\cup\{0\}$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, $\xi_k \neq 0$, $1 \leq k \leq n$. Then*

$$(\|x\|_p^\beta)^\wedge(\xi) = ((|x_1|^p + \cdots + |x_n|^p)^{\beta/p})^\wedge(\xi) = \frac{p}{\Gamma(-\beta/p)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_p(t\xi_k)dt.$$

*Proof:* Assume that $-1 < \beta < 0$. By the definition of the $\Gamma$-function

$$(|x_1|^p + \cdots + |x_n|^p)^{\beta/p} = \frac{p}{\Gamma(-\beta/p)} \int_0^\infty y^{-1-\beta} \exp(-y^p(|x_1|^p + \cdots + |x_n|^p))dy.$$

For every fixed $y > 0$, the Fourier transform of the function

$$x \to \exp(-y^p(|x_1|^p + \cdots + |x_n|^p))$$

at any point $\xi \in \mathbb{R}^n$ is equal to $y^{-n}\prod_{k=1}^n \gamma_p(\xi_k/y)$. Making the change of variables $t = 1/y$ we get

$$\begin{aligned} ((|x_1|^p + \cdots + |x_n|^p)^{\beta/p})^\wedge(\xi) &= \frac{p}{\Gamma(-\beta/p)} \int_0^\infty y^{-n-\beta-1} \prod_{k=1}^n \gamma_p(\xi_k/y)dy \\ &= \frac{p}{\Gamma(-\beta/p)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_p(t\xi_k)dt. \end{aligned} \tag{4}$$

The latter integral converges if $-n < \beta < pn$ since the function $t \to \prod_{k=1}^n \gamma_p(t\xi_k)$ decreases at infinity like $t^{-n-np}$ (recall that $\xi_k \neq 0$, $1 \leq k \leq n$).

If $\beta$ is allowed to assume complex values, then both sides of (4) are analytic functions of $\beta$ in the domain $\{-n < \operatorname{Re}\beta < np, \beta/p \notin \mathbb{N} \cup \{0\}\}$. These two functions admit unique analytic continuation from the interval $(-1, 0)$. Thus the equality (4) remains valid for all $\beta \in (-n, pn)$, $\beta/p \notin \mathbb{N} \cup \{0\}$ (see [3] for details of analytic continuation in such situations). ∎

Now we can use Lemma 2 with $\beta = -n+1$ and Theorem 1 to get an expression for the volume of central sections. Note that the condition of Lemma 2 that $\xi$ has non-zero coordinates may be removed from Corollary 1 because the volume is a continuous function of $\xi$. In the case where $1 \leq p \leq 2$, the result of Corollary 1 was proved by Meyer and Pajor [7].

COROLLARY 1: *For every $p > 0$ and $\xi \in \Omega$,*

$$\mathrm{Vol}_{n-1}(B_p \cap \xi^\perp) = \frac{p}{\pi(n-1)\Gamma((n-1)/p)} \int_0^\infty \prod_{k=1}^n \gamma_p(t\xi_k)dt.$$

## 3. Minimal sections of the $\ell_p^n$-balls, $0 < p < 2$

We need the following property of the functions $\gamma_p$ with $0 < p < 2$.

LEMMA 3: *For every $p \in (0,2)$, the function $\gamma_p(\sqrt{t})$ is log-convex on $(0,\infty)$. In other words, the function $\gamma_p'(t)/(t\gamma_p(t))$ is increasing on $(0,\infty)$. Also, for every $k,m \in \mathbb{N}, k < m$ and every $t > 0$, we have $\gamma_p^k(k^{-1/2}t)\gamma_p^{m-k}(0) \geq \gamma_p^m(m^{-1/2}t)$.*

*Proof:* A well-known fact is that there exists a measure $\mu$ on $[0,\infty)$ whose Laplace transform is equal to $\exp(-t^{p/2})$. This is a stable measure, and its properties and asymptotic behavior of its density (which decreases at infinity as $|t|^{-1-p/2}$, up to a constant) are described, for example, in [8]. For every $z \in \mathbb{R}$, we have

$$\exp(-|z|^p) = \int_0^\infty \exp(-uz^2)d\mu(u).$$

Calculating the Fourier transform of both sides of the latter equality as functions of the variable $z$, we get, for every $t \in \mathbb{R}$,

$$\gamma_p(t) = \sqrt{2\pi}\int_0^\infty u^{-1/2}\exp\left(\frac{-t^2}{4u}\right)d\mu(u),$$

where the integral converges because of the asymptotics of the density of $\mu$ at infinity, as mentioned above. Now the fact that

$$\gamma_p^2(\sqrt{(t_1+t_2)/2}) \leq \gamma_p(\sqrt{t_1})\gamma_p(\sqrt{t_2})$$

follows from the Cauchy-Schwarz inequality applied to the functions $\exp(-t_1/(8u))$ and $\exp(-t_2/(8u))$ and the measure $u^{-1/2}d\mu(u)$, where $t_1, t_2$ are arbitrary positive numbers. Therefore, the function $\gamma_p(\sqrt{t})$ is log-convex which implies the other two statements of Lemma 3. ∎

Now we are ready to find the minimal sections of $\ell_p^n$-balls.

THEOREM 2: *For every $p \in (0,2)$ and every $\xi \in \Omega$,*

$$\frac{p}{\pi(n-1)\Gamma((n-1)/p)}\int_0^\infty \gamma_p^n(t/\sqrt{n})dt \leq \mathrm{Vol}_{n-1}(B_p \cap \xi^\perp)$$

$$\leq \frac{p}{\pi(n-1)\Gamma((n-1)/p)}\gamma_p^{n-1}(0)\int_0^\infty \gamma_p(t)dt = \frac{2^{n-1}p(\Gamma(1+1/p))^{n-1}}{(n-1)\Gamma((n-1)/p)}$$

*with the left inequality turning into an equality if and only if $|\xi_i| = 1/\sqrt{n}$ for every $i$, and the upper bound occurs if and only if one of the coordinates of the vector $\xi$ is equal to $\pm 1$ and the others are equal to zero.*

*Proof:* Consider the function

$$F(\xi_1,\ldots,\xi_n) = \int_0^\infty \gamma_p(t\xi_1)\cdots\gamma_p(t\xi_n)dt + \lambda(\xi_1^2+\cdots+\xi_n^2-1),$$

where $\lambda$ is the Lagrange multiplier. It suffices to find the maximal and minimal value of the function $F$ in the positive octant under the condition $\xi_1^2+\cdots+\xi_n^2=1$. To find the critical points of the function $F$, we have to solve the system of equations

$$\frac{\partial F}{\partial \xi_i}(\xi)=\int_0^\infty t\gamma_p'(t\xi_i)\prod_{k\neq i}\gamma_p(t\xi_k)dt+2\lambda\xi_i=0,$$

where $i=1,\ldots,n$. For each $i$ with $\xi_i\neq 0$, we can write the latter equality in the following form:

$$\int_0^\infty t\frac{\gamma_p'(t\xi_i)}{\xi_i\gamma_p(t\xi_i)}\prod_{k=1}^n\gamma_p(t\xi_k)dt=-2\lambda. \tag{5}$$

Since (by Lemma 3) the function $\gamma_p'(t\xi_i)/(\xi_i\gamma_p(t\xi_i))$ is increasing and $\gamma_p$ is non-negative, we can have (5) for different values of $i$ simultaneously only if the corresponding coordinates of the vector $\xi$ are equal. Therefore, the critical points of the function $F$ are only those points $\xi$ for which some of the coordinates are zero, and the absolute values of the rest are equal. Hence, the problem is reduced to comparing the values of $F$ at the points $\xi^{(k)}, k=1,\ldots,n$, where the first $k$ coordinates of $\xi^{(k)}$ are equal to $1/\sqrt{k}$ and the last $n-k$ coordinates are equal to zero. It follows from the inequality of Lemma 3 that the maximal value of $F$ on the sphere $\Omega$ occurs at the point $\xi^{(1)}$, and the minimal value is at the point $\xi^{(n)}$. Now the result of Theorem 2 follows from Corollary 1. ∎

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## References

[1] K. Ball, *Cube slicing in* $\mathbb{R}^n$, Proceedings of the American Mathematical Society **97** (1986), 465–473.

[2] R. J. Gardner, *Geometric Tomography*, Cambridge University Press, Cambridge, 1995.

[3] I. M. Gelfand and G. E. Shilov, *Generalized Functions 1. Properties and Operations*, Academic Press, New York, 1964.

[4] S. Helgason, *The Radon Transform*, Birkhäuser, Boston, 1980.

[5] A. Koldobsky, *Schoenberg's problem on positive definite functions* (English translation in St. Petersburg Mathematical Journal **3** (1992), 563–570), Algebra and Analysis **3** (1991), 78–85.

[6] A. Koldobsky, *Characterization of measures by potentials*, Journal of Theoretical Probability **7** (1994), 135–145.

[7] M. Meyer and A. Pajor, *Sections of the unit ball of $\ell_p^n$*, Journal of Functional Analysis **80** (1988), 109–123.

[8] V. M. Zolotarev, *One-Dimensional Stable Distributions*, American Mathematical Society, Providence, 1986.